Thawing quintessence with a nearly flat potential

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The thawing quintessence model with a nearly flat potential provides a natural mechanism to produce an equation of state parameter, w, close to -1 today. We examine the behavior of such models for the case in which the potential satisfies the slow roll conditions: $[(1/V)(dV/d\phi)]^2 \ll 1$ and $(1/V)(d^2V/d\phi^2) \ll 1$, and we derive the analog of the slow-roll approximation for the case in which both matter and a scalar field contribute to the density. We show that in this limit, all such models converge to a unique relation between $1+w$, Ω_ϕ, and the initial value of $(1/V)(dV/d\phi)$. We derive this relation, and use it to determine the corresponding expression for $w(a)$, which depends only on the present-day values for w and Ω_ϕ. For a variety of potentials, our limiting expression for $w(a)$ is typically accurate to within $\delta w \lesssim 0.005$ for $w < -0.9$. For redshift $z \lesssim 1$, $w(a)$ is well-fit by the Chevallier-Polarski-Linder parametrization, in which $w(a)$ is a linear function of a.

I. INTRODUCTION

Observational evidence [1, 2] indicates that roughly 70% of the energy density in the universe is in the form of an exotic, negative-pressure component, dubbed dark energy. (See Ref. [3] for a recent review). The observational bounds on the properties of the dark energy have continued to tighten. Taking w to be the ratio of pressure to density for the dark energy:

$$w = p_{DE}/\rho_{DE}, \qquad (1)$$

recent observational constraints are typically $-1.1 \lesssim w \lesssim -0.9$ when w is assumed constant (see, e.g., [4, 5] and references therein).

We can consider two possibilities. If the measured value of w continues to converge to a value arbitrarily close to -1, then it is most reasonable to assume a cosmological constant (a conclusion supported by both the Akaike information criterion [6] and common sense). On the other hand, it is conceivable that the observations will converge on a value of w very close to, but not exactly equal to, -1. In this case, we must consider how such a dark energy equation of state might arise.

One possibility, dubbed quintessence, is a model in which the dark energy arises from a scalar field [7, 8, 9, 10, 11]. Caldwell and Linder [12] showed that quintessence models in which the scalar field potential asymptotically approaches zero can be divided naturally into two categories, which they dubbed "freezing" and "thawing" models, with quite different behaviors. Thawing models have a value of w which begins near -1 and increases with time, while freezing models have a value of w which decreases with time, with an asymptotic value that depends on the shape of the potential. (If the observations converge to a value of w less than -1, more exotic models must be considered. We will not consider this possibility here).

Thawing models with a nearly flat potential provide a natural way to produce a value of w that is close to,

but not exactly equal to -1, since the field begins with $w \approx -1$, and w increases only slightly up to the present. Furthermore, with a nearly flat potential and $w \approx -1$, the dynamics of quintessence are considerably simplified. In this paper, we show that all such models converge to a single unique evolution. In addition to providing a plausible model for w near -1, such models (since they all converge to a single type of behavior) can serve as a useful set of fiducial models that can be compared to ΛCDM.

In the next section, we reexamine thawing quintessence models and outline the arguments for these models. In Sec. 3, we present a general description of the behavior of such models. Our main results are given in equations (23) and (26). In Sec. 4, we discuss the arguments against thawing quintessence models. Our conclusions are summarized briefly in Sec. 5.

II. THE CASE FOR THAWING QUINTESSENCE

We will assume that the dark energy is provided by a minimally-coupled scalar field, ϕ, with equation of motion given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \qquad (2)$$

where the Hubble parameter H is given by

$$H = \left(\frac{\dot{a}}{a}\right) = \sqrt{\rho/3}. \qquad (3)$$

Here a is the scale factor, ρ is the total density, and we work in units for which $8\pi G = 1$. Equation (2) indicates that the field rolls downhill in the potential $V(\phi)$, but its motion is damped by a term proportional to H.

The pressure and density of the scalar field are given

by

$$p = \frac{\dot{\phi}^2}{2} - V(\phi), \qquad (4)$$

and

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi), \qquad (5)$$

respectively, and the equation of state parameter, w, is given by equation (1).

As noted above, observations suggest a value of w near -1. Caldwell and Linder [12] noted there are basically two ways to achieve such a result (see also Ref. [13] for a more detailed discussion of some of these issues). In the first case, freezing potentials, the field is initially rolling down the potential with $w \neq -1$, but it slows with time, driving w toward -1. (As emphasized in Ref. [13], the tracking models introduced in Ref. [11] are a subset of freezing models, but not all freezing models display tracking behavior). In thawing models, the field is initially nearly frozen at some value ϕ_0, with $w = -1$. Then as H decreases, the field rolls down the potential, and w increases with time.

It is possible to produce w near -1 at the present using either freezing or thawing models. Chongchitnan and Efstathiou [14] used Monte Carlo simulations to derive a set of potentials that yield w very close to -1 today. They found two classes of acceptable solutions: very flat potentials, and models in which the field evolves from a region with very steep slope in the potential to a region in which the potential is roughly flat. While neither type of model can be ruled out, we feel that the models with a nearly flat potential are clearly a more natural way to produce the desired present-day value of w near -1.

A similar naturalness issue was raised by Bludman [16], who argued that the only way to achieve tracking models with w near -1 today was for the model to contain a sharp change in the curvature of the potential at the present. Thus, we are presented with a double coincidence problem: why should the field be entering this special region of the potential at the same time that the dark energy is coming to dominate the matter, and why are both of these happening right now? In thawing models with a nearly flat potential, on the other hand, w never deviates very far from -1.

Another argument in favor of these thawing models is that we already have strong evidence that the universe at one time underwent a period of vacuum energy domination (inflation). Many models for inflation correspond to the sort of thawing models examined here [15]; the scalar field initially has $w = -1$, but then rolls downhill to terminate inflation.

Finally, Griest [17] has suggested a solution to the coincidence problem involving thawing fields. In his model, the universe contains a variety of scalar fields with various initial energy scales $V_i(\phi_0)$, $i = 1, 2, 3, ...$ As the matter or radiation density drops below a given $V_i(\phi_0)$, the universe undergoes a period of dark energy domination, but the field then thaws and slides down the potential, allowing matter or radiation to dominate again. Given enough of these fields, it would not be suprising to find ourselves in an epoch in which one of them is just beginning to dominate at present (see also the somewhat different model of Ref. [18]).

None of these arguments proves, of course, that a universe with w close to -1 (but not equal to -1) must involve a thawing quintessence field with a nearly flat potential. However, they do indicate that such models are worthy of further study.

III. EVOLUTION OF THAWING QUINTESSENCE WITH NEARLY FLAT POTENTIAL

We will assume a scalar field with initial value ϕ_0 in a nearly flat potential $V(\phi)$. Specifically, we will assume that at $\phi = \phi_0$, the field satisfies the slow-roll conditions:

$$\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \ll 1, \qquad (6)$$

and

$$\frac{1}{V}\frac{d^2V}{d\phi^2} \ll 1. \qquad (7)$$

The latter condition corresponds to a mass scale $m_\phi \ll \sqrt{V(\phi_0)}$. Setting $V(\phi_0)$ roughly equal to the dark energy density at the present, we get $m_\phi \ll 10^{-33}$ eV. This is the same (unnaturally small) mass that occurs generically in quintessence models for dark energy.

In analyzing models for inflation, it is usually assumed that the scalar field dominates the expansion, and that $V(\phi) \gg \dot{\phi}^2/2$. With these assumptions, along with the flatness of the potential given by equations (6) and (7), it can be shown that the $\ddot{\phi}$ term in equation (2) can be neglected, yielding the simple equation $3H\dot{\phi} = -dV/d\phi$ (see, e.g., Ref. [15]). This is called the slow-roll approximation.

It is well-known that the slow-roll approximation fails for the case of quintessence [13, 16, 19]. The basic reason is that the slow-roll approximation requires the scalar field to dominate the expansion. However, this is never the case for quintessence, since matter always contributes significantly to the total density. However, nothing prevents us from assuming the slow roll conditions on the potential (equations 6 and 7) along with the requirement that w be close to -1 today. Effectively, we are deriving the analog of the slow-roll approximation for the case where the expansion is not dominated by the scalar field.

At the late times which are of interest here, the universe is dominated by dark energy (assumed to arise from a scalar field) and nonrelativistic matter; we can neglect the radiation component. We assume a flat universe containing only matter and a scalar field, so that

$\Omega_\phi + \Omega_M = 1$. Then equations (2) and (3) can be rewritten in terms of the variables x, y, and λ, defined by

$$x = \phi'/\sqrt{6}, \tag{8}$$
$$y = \sqrt{V(\phi)/3H^2}, \tag{9}$$
$$\lambda = -\frac{1}{V}\frac{dV}{d\phi}, \tag{10}$$

and the prime will always denote the derivative with respect to $\ln a$: e.g., $\phi' \equiv a(d\phi/da)$. (This discussion, from equation 8 through equation 16, is taken from Refs. [20, 21, 22]).

Then x^2 gives the contribution of the kinetic energy of the scalar field to Ω_ϕ, and y^2 gives the contribution of the potential energy, so that

$$\Omega_\phi = x^2 + y^2, \tag{11}$$

while the equation of state is

$$\gamma \equiv 1 + w = \frac{2x^2}{x^2 + y^2}. \tag{12}$$

It is convenient to work in terms of γ, since we are interested in models for which w is near -1, so γ is near zero, and we can then expand quantities of interest to lowest order in γ. Equations (2) and (3), in a universe containing only matter and a scalar field, become

$$x' = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x[1 + x^2 - y^2], \tag{13}$$

$$y' = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y[1 + x^2 - y^2], \tag{14}$$

$$\lambda' = -\sqrt{6}\lambda^2(\Gamma - 1)x, \tag{15}$$

where

$$\Gamma \equiv V\frac{d^2V}{d\phi^2}\bigg/\left(\frac{dV}{d\phi}\right)^2. \tag{16}$$

We now rewrite these equations, changing the dependent variables from x and y to the observable quantities Ω_ϕ and γ given by equations (11) and (12). To make this transformation, we assume that $x' > 0$; our results generalize trivially to the opposite case. We obtain:

$$\gamma' = -3\gamma(2 - \gamma) + \lambda(2 - \gamma)\sqrt{3\gamma\Omega_\phi}, \tag{17}$$
$$\Omega'_\phi = 3(1 - \gamma)\Omega_\phi(1 - \Omega_\phi), \tag{18}$$
$$\lambda' = -\sqrt{3}\lambda^2(\Gamma - 1)\sqrt{\gamma\Omega_\phi}. \tag{19}$$

Note that equation (17) also follows, in a trivial way, from the expression for w' given in Ref. [13]. Finally, we will see that the equations simplify if we transform our dependent variable from a to $\Omega_\phi(a)$. This gives us

$$\frac{d\gamma}{d\Omega_\phi} = \frac{\gamma'}{\Omega'_\phi} = \frac{-3\gamma(2 - \gamma) + \lambda(2 - \gamma)\sqrt{3\gamma\Omega_\phi}}{3(1 - \gamma)\Omega_\phi(1 - \Omega_\phi)}. \tag{20}$$

This change of variables is valid only if Ω_ϕ is a monotonic function of the scale factor; it breaks down at any point where $d\Omega_\phi/da = 0$. This condition is satisfied for most quintessence models and for all of the models we consider here; it is not satisfied, for example, in models in which Ω_ϕ oscillates in time [18].

Equations (19) and (20) are an exact description of the scalar field evolution for $x' > 0$, but they do not yield any simple solution. At this point, we make two assumptions. Our first assumption is that $\gamma \ll 1$, corresponding to w near -1. The second assumption is that λ is approximately constant, so that

$$\lambda = \lambda_0 = -(1/V)(dV/d\phi)\bigg|_{\phi=\phi_0}, \tag{21}$$

i.e., λ_0 is the value of λ at the initial value of the scalar field ϕ_0 before it begins to roll down the potential. Equation (21) follows from the slow-roll conditions, equations (6) and (7), as we will show at the end of this calculation. Replacing λ with λ_0 and retaining terms to lowest order in γ in equation (20) yields the following:

$$\frac{d\gamma}{d\Omega_\phi} = -\frac{2\gamma}{\Omega_\phi(1 - \Omega_\phi)} + \frac{2}{3}\lambda_0\frac{\sqrt{3\gamma}}{(1 - \Omega_\phi)\sqrt{\Omega_\phi}}. \tag{22}$$

This equation can be transformed into a linear differential equation with the change of variables $s^2 = \gamma$, and the resulting equation can be solved exactly. For the models of interest here, we have the boundary condition $\gamma = 0$ at $\Omega_\phi = 0$. The resulting solution (reexpressed in terms of w) is

$$1 + w = \frac{\lambda_0^2}{3}\left[\frac{1}{\sqrt{\Omega_\phi}} - \left(\frac{1}{\Omega_\phi} - 1\right)\tanh^{-1}\sqrt{\Omega_\phi}\right]^2,$$
$$= \frac{\lambda_0^2}{3}\left[\frac{1}{\sqrt{\Omega_\phi}} - \frac{1}{2}\left(\frac{1}{\Omega_\phi} - 1\right)\ln\left(\frac{1 + \sqrt{\Omega_\phi}}{1 - \sqrt{\Omega_\phi}}\right)\right]^2 \tag{23}$$

Equation (23), along with the corresponding result for $w(a)$ derived below, is our main result. It shows that for sufficiently flat potentials, all thawing quintessence models with w near -1 approach a single generic behavior, with $w(a)$ determined entirely by $\Omega_\phi(a)$ and the (constant) initial value of $(1/V)(dV/d\phi)$. A graph of this generic relationship between w, Ω_ϕ, and λ_0 is given in Fig. 1.

Equation (23) shows that $1 + w \sim O(\lambda_0^2)$. Thus, our first slow-roll condition (equation 6) insures that $1 + w \ll 1$, as desired. The condition that λ be nearly constant up to the present day can be quantified by requiring $|\lambda'/\lambda| \ll 1$. Taking γ to be of order λ^2 in equation (19), we obtain the condition

$$\frac{1}{V}\frac{d^2V}{d\phi^2} - \left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \ll 1. \tag{24}$$

The two slow-roll conditions, taken together, insure that this condition is satisfied.
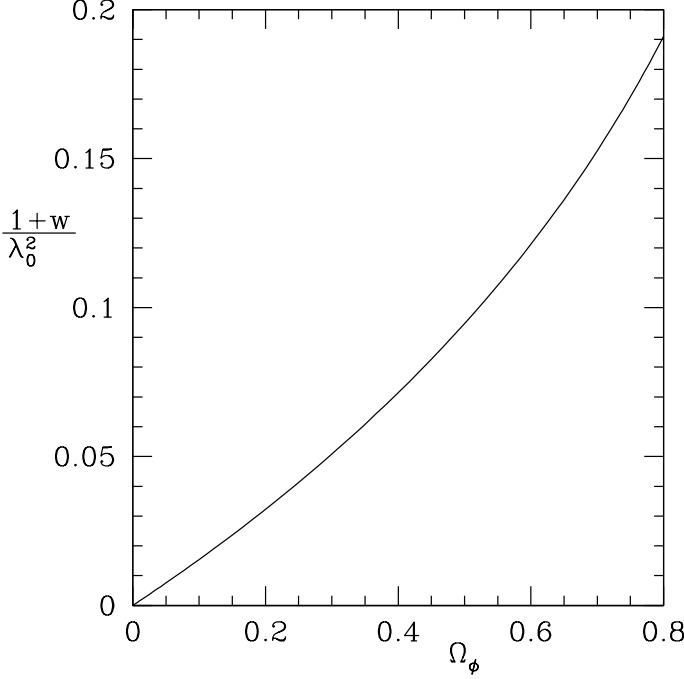
FIG. 1: The value of $(1+w)/\lambda_0^2$ as a function of Ω_ϕ in thawing quintessence models with a nearly flat potential and w near -1. Here w and Ω_ϕ are functions of the scale factor a, while λ_0 is the (constant) initial value of $-(1/V)(dV/d\phi)$.
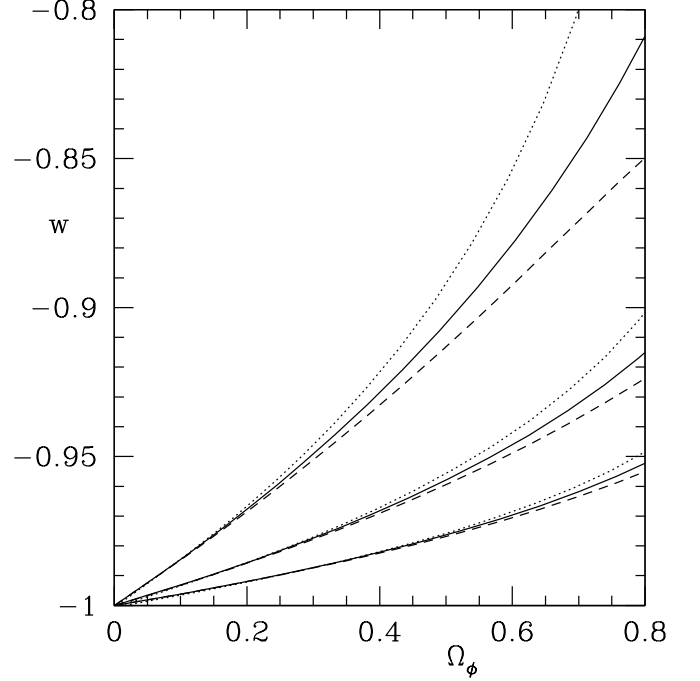

FIG. 2: A comparison between $w(\Omega_\phi)$ for $V(\phi) = \phi^2$ (dotted curve), $V(\phi) = \phi^{-2}$ (dashed curve) and our analytic result for $w(\Omega_\phi)$ (solid curve) for (top to bottom) $\lambda_0 = 1$, $\lambda_0 = 2/3$, and $\lambda_0 = 1/2$.

A sufficiently accurate determination of the present-day values of w and Ω_ϕ (w_0 and $\Omega_{\phi0}$, respectively) uniquely determines the value of λ_0 for these models. For example, for $\Omega_{\phi0} = 0.7$ and $w_0 = -0.9$, we obtain $\lambda_0 = 0.8$.

The way in which arbitrary potentials satisfying the slow-roll conditions converge to equation (23) is illustrated in Fig. 2. In this figure, the solid curve gives the behavior for $w(\Omega_\phi)$ predicted by equation (23), while the dotted and dashed curves give the true evolution for the potentials $V = \phi^2$ and $V = \phi^{-2}$, respectively, where we choose the initial value of ϕ such that $\lambda_0 = 1$, $2/3$, $1/2$. As expected, agreement is poor for $\lambda_0 = 1$ and improves for smaller values of λ_0. Since we dropped terms of order $1 + w$ in deriving equation (23), we expect the fractional error in $1 + w$ to be on the order of $1 + w$. For $w < -0.9$, this translates into an error in w of $\delta w \lesssim 0.01$, which is apparent in Fig. 2.

The behavior of the ϕ^{-2} potential demonstrates an important point: while negative power law potentials usually give rise to "freezing" models [7, 10, 11], they can be made to act as thawing models by an appropriate choice of ϕ_0. For example, when $V(\phi) = \phi^{-n}$, if $\phi_0 \gg n$, then equations (6) and (7) are satisfied, and the model behaves like a thawing model. This shows that *any* potential can give rise to the type of models discussed here, as long as $V(\phi)$ has a region over which equations (6) and (7) apply.

We can use equation (18) to solve for Ω_ϕ as a function

of a and thus determine $w(a)$. Taking the limit $\gamma \ll 1$ in equation (18) gives $\Omega'_\phi = 3\Omega_\phi(1 - \Omega_\phi)$, with solution

$$\Omega_\phi = \left[1 + \left(\Omega_{\phi0}^{-1} - 1\right) a^{-3}\right]^{-1}, \qquad (25)$$

where $\Omega_{\phi0}$ is the present-day value of Ω_ϕ, and we take $a = 1$ at the present. Equation (25) is identical to the expression for Ω_Λ as a function of a in the ΛCDM model, which is not surprising, as we are taking w near -1 (see also Ref. [23]). Equations (23) and (25) together give an explicit expression for w as a function of a. Assuming a particular value w_0 for the present-day value of w, we can then eliminate λ_0 from this expression, so that $w(a)$ is a unique function of w_0 and $\Omega_{\phi0}$. We obtain:

$$1 + w = (1 + w_0)\left[\sqrt{1 + (\Omega_{\phi0}^{-1} - 1)a^{-3}}\right.$$

$$\left. -(\Omega_{\phi0}^{-1} - 1)a^{-3}\tanh^{-1}\frac{1}{\sqrt{1 + (\Omega_{\phi0}^{-1} - 1)a^{-3}}}\right]^2$$

$$\times \left[\frac{1}{\sqrt{\Omega_{\phi0}}} - \left(\frac{1}{\Omega_{\phi0}} - 1\right)\tanh^{-1}\sqrt{\Omega_{\phi0}}\right]^{-2}. \quad (26)$$

A graph of $w(a)$ is shown in Fig. 3 for several values of w_0 and $\Omega_{\phi0}$. Note that $w(a)$ depends primarily on w_0 and is not very sensitive to the value of $\Omega_{\phi0}$ for $0.6 < \Omega_{\phi0} < 0.8$.
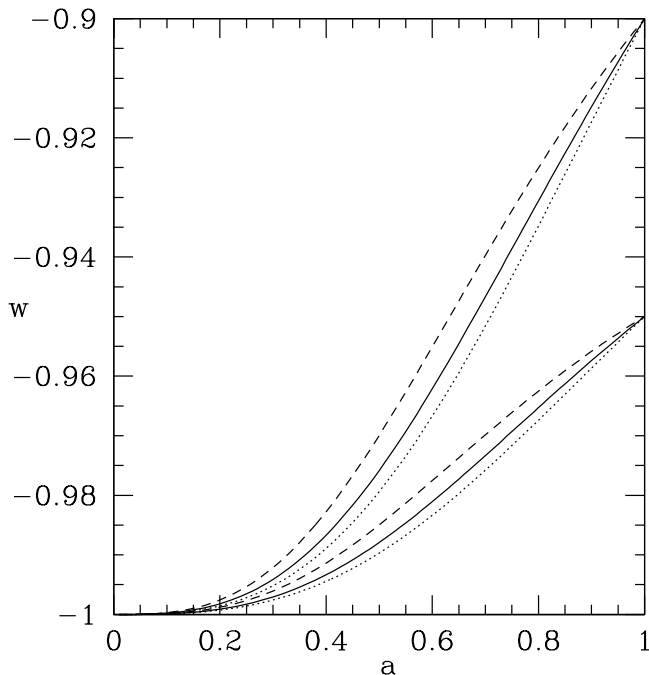
FIG. 3: Our analytic result for the evolution of w as a function of the scale factor, a, normalized to $a = 1$ at the present, in thawing quintessence models with a nearly flat potential and w near -1, for $\Omega_{\phi 0} = 0.8$ (dashed), $\Omega_{\phi 0} = 0.7$ (solid) and $\Omega_{\phi 0} = 0.6$ (dotted). Upper three curves are for $w_0 = -0.9$; lower three curves are for $w_0 = -0.95$.
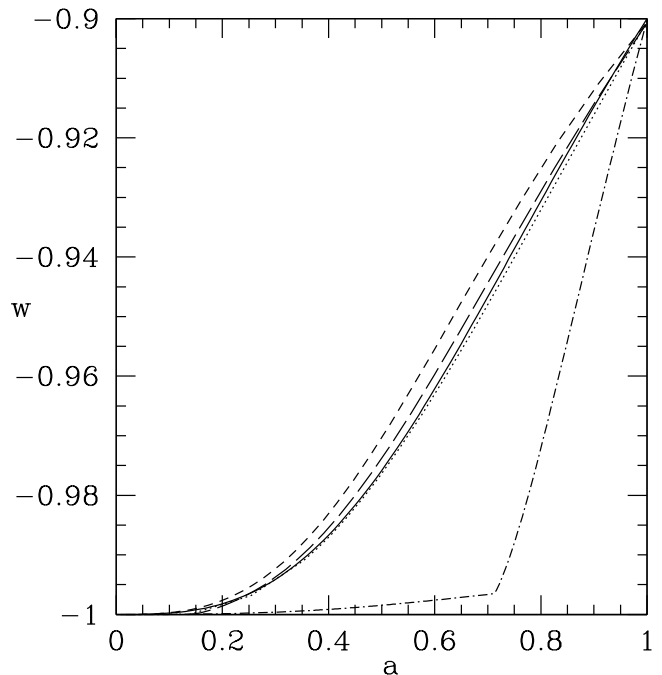
FIG. 4: The evolution of w as a function of the scale factor, a, normalized to $a = 1$ at the present, with $\Omega_{\phi 0} = 0.7$ and $w_0 = -0.9$. Solid curve is our analytic result for the behavior of thawing models with a nearly flat potential and w near -1. Other curves give the true evolution for the potentials $V(\phi) = \phi^2$ (dotted), $V(\phi) = \phi^{-2}$ (short dash), and $V(\phi) = \exp(-\lambda\phi)$ (long dash). Dot-dash curve is a model with $V(\phi) = \exp(-\lambda\phi)$ in which λ changes discontinuously.

Also, $w(a)$ is a nearly linear function of a for a between 0.5 and 1 (redshift $z \lesssim 1$), although the linear behavior breaks down for $z > 1$. The $z \lesssim 1$ behavior agrees well with the Chevallier-Polarski-Linder parametrization [24, 25], in which $w(a)$ is taken to have the form $w(a) = w_0 + w_a(1 - a)$, although equation (26) does not provide any particular insight into the origin of this linear behavior. However, in our case w_a is not a free parameter, but depends on w_0 and $\Omega_{\phi 0}$; for a fixed value of $\Omega_{\phi 0}$, equation (26) corresponds to a one-parameter family of models. For instance, for $\Omega_{\phi 0} = 0.7$, the linear fit to equation (26) for $z \lesssim 1$ is roughly $w = w_0 - 1.5(1 + w_0)(1 - a)$, so that $w_a \approx -1.5(1 + w_0)$.

Now we can compare the generic behavior predicted by equation (26) with the actual scalar field evolution. In Fig. 4, we show this predicted behavior, along with $w(a)$ for the potentials $V = \phi^2$, $V = \phi^{-2}$, and $V = \exp(-\lambda\phi)$. The value of ϕ_0 for the power law potentials is chosen to give $\Omega_{\phi 0} = 0.7$ and $w_0 = -0.9$. For the exponential potential, $\Omega_{\phi 0} = 0.7$ and $w_0 = -0.9$ are fixed by the value of λ. The typical errors here are $\delta w \lesssim 0.005$, showing strong agreement between the true evolution of $w(a)$ and our analytic expression for $w(a)$. The error decreases as w_0 decreases, so Fig. 4 gives an upper limit on the error in our approximation for $w_0 < -0.9$.

In Fig. 5, we compare our limiting behavior for $w(a)$ in

equation (26) to the SNIa observations. The likelihoods were constructed using the 60 Essence supernovae, 57 SNLS (Supernova Legacy Survey) and 45 nearby supernovae, and the new data release of 30 SNe Ia detected by HST and classified as the Gold sample by Riess et al. [2, 26]. The combined dataset can be found in Ref. [5]. It is clear that current observations do not exclude the thawing quintessence models we have considered here, although the observations are also obviously consistent with a cosmological constant. The maximum likelihood point actually lies below $w_0 = -1$, but we have not extended our graph to $w_0 < -1$, as our derivation of equation (26) assumes a standard quintessence model with $w \geq -1$ at all times.

We have shown that the slow-roll conditions, equations (6) and (7), are sufficient to allow a thawing model to produce w near -1 today, but are they also necessary conditions? The answer is no, although violating these bounds with a thawing model requires rather unusual potentials. Following Ref. [14], we have produced a potential by joining the functions $V(\phi) = \exp(-\lambda_1\phi)$ and $V(\phi) = \exp(-\lambda_2\phi)$ at $\phi = 0$. The potential is then continuous, but λ varies discontinuously, so equation (7) is violated. In Fig. 4, we show $w(a)$ for this thawing potential (dot-dash curve), where we have taken $\lambda_1 = 0.2$,
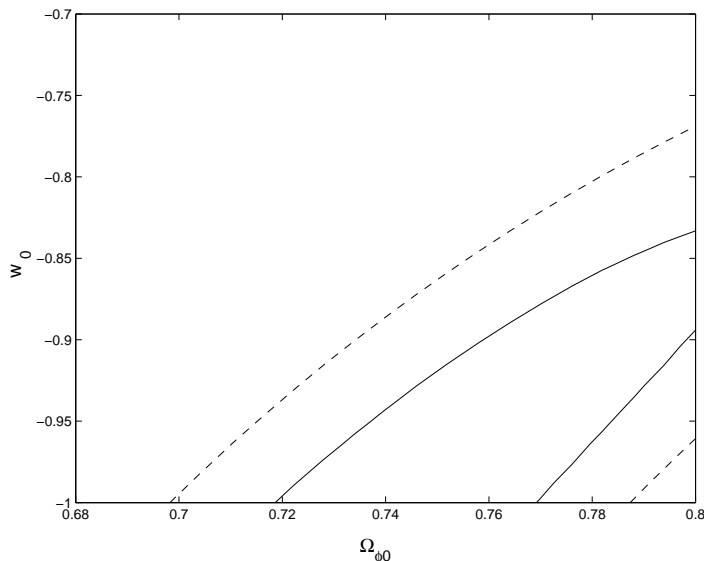
FIG. 5: The $1-\sigma$ (solid) and $2-\sigma$ (dashed) contours in the plane defined by the present-day values of Ω_ϕ and w for the quintessence field (denoted $\Omega_{\phi 0}$ and w_0 respectively), for the thawing behavior given by equation (26).

$\lambda_2 = 1.04$, and we have chosen ϕ_0 to give $w_0 = -0.9$ and $\Omega_{\phi 0} = 0.7$. The evolution of ϕ is clearly always in the thawing regime ($dw/da > 0$), and it gives w near -1, but it does not produce a functional form for $w(a)$ resembling that of a nearly flat potential. Of course, this form for the potential is rather pathological (note that it also violates the upper bound on w' as a function of w for thawing models postulated in Ref. [12]).

One of the best-motivated thawing quintessence models is the Pseudo-Nambu Goldstone Boson (PNGB) model [27]. (For a recent discussion, see Ref. [28] and references therein). This model is characterized by the potential

$$V(\phi) = M^4[\cos(\phi/f) + 1], \qquad (27)$$

and ϕ_0 can be taken to lie between 0 and πf. Then the evolution of this model is a function of M, f, and ϕ_0. Using equations (6) and (7), we see that the slow-roll conditions are satisfied for all ϕ_0 if $f > 1$, while they cannot be satisfied for any ϕ_0 if $f < 1$. The latter result follows from the trigonometic function in the PNGB potential: its second derivative is large whenever the first derivative is small, and vice-versa. None of these results depend on the value of M. Thus, our results are a good approximation to the behavior of the PNGB model for the case where $f > 1$.

Now we consider some related approximation schemes. Crittenden et al. [23] analyzed quintessence models with w near -1 in terms of the parameter $\kappa(\phi)$, defined through the equation

$$\kappa(\phi) = \frac{dV/d\phi}{V(1 + \ddot{\phi}/3H\dot{\phi})}. \qquad (28)$$

With w near -1, they took the evolution for Ω_ϕ to be given by equation (25), and they approximated the evolution of κ as a linear function of ϕ:

$$\kappa(\phi) = \kappa_0 + \kappa_1(\phi - \phi_0). \qquad (29)$$

Thus, the model of Ref. [23] has two free parameters, κ_0 and κ_1, which determine $w(a)$. It is straightfoward to derive the equivalent of our equations (23) and (26). For w as a function of Ω_ϕ, we obtain:

$$1 + w = \frac{2}{3}\kappa_0^2 \Omega_\phi \left(\frac{1 - \Omega_\phi}{1 - \Omega_{\phi 0}}\right)^{4\kappa_1/3}. \qquad (30)$$

Our corresponding result (equation 23) is clearly distinct from this result for all values of κ_0 and κ_1. Note further that equation (30) implies $1 + w_0 = (2/3)\kappa_0^2 \Omega_{\phi 0}$, so we can express w as a function of w_0, $\Omega_{\phi 0}$, and κ_1 alone (corresponding to equation 26); κ_0 drops out of this expression:

$$1 + w = (1 + w_0)a^3 \left[\Omega_{\phi 0}a^3 + (1 - \Omega_{\phi 0})\right]^{-(4\kappa_1 + 3)/3}. \qquad (31)$$

Again, this result is distinct from equation (26), although the two expressions obviously can be made to converge to similar behavior by the appropriate choice of κ_1, since κ_1 can be chosen to give good agreement with the exact evolution [23]. The main difference between this approach and ours is that our final result for $w(a)$ contains no free parameters; it is a function only of w_0 and $\Omega_{\phi 0}$, while the expression for $w(a)$ in the form of equation (31) contains the fitting parameter κ_1.

Neupane and Scherer [29] considered the consequences of fixing x (as defined in equation 8) to be a constant, α. With $x = \alpha$, their relation corresponding to our equation (23) is

$$1 + w = \frac{\alpha^2}{3\Omega_\phi}. \qquad (32)$$

While it is possible in such models to produce w close to -1 at the present, equation (32) shows that these $w \approx -1$ models always act as freezing models, since increasing Ω_ϕ corresponds to decreasing $1 + w$.

One might argue that the correct "generic" model for a nearly flat potential should be a linear potential, i.e., constant $dV/d\phi$, rather than constant $(1/V)(dV/d\phi)$. Linear potentials have been investigated previously by a number of authors [10, 30, 31, 32, 33, 34]. Exact solutions for the linear potential have been derived for the case where the universe is scalar field dominated [30] or when it is matter-dominated [10], but not for the intermediate case. However, it is possible to use the techniques discussed here to derive an approximate solution. If we take

$$V = V_0 - \alpha\phi, \qquad (33)$$

then equation (2) has the solution

$$\dot{\phi} = \alpha \int_{t=t_i}^{t_f} \left[\frac{a(t)}{a(t_f)}\right]^3 dt. \qquad (34)$$

The integrand can be reexpressed in terms of a and $H(a)$ to give

$$\dot{\phi} = \alpha \int_{a=a_i}^{a_f} \frac{1}{H(a)} \left(\frac{a}{a_f}\right)^3 \frac{da}{a}. \qquad (35)$$

This solution is as yet exact. Now we make essentially the same approximations that we used earlier for $1+w \ll 1$. We take $H(a)$ to be given by equation (3), but in determining the total value of ρ, we approximate ρ_ϕ as a constant, and we take $1+w$ to be given by $1+w \approx \dot{\phi}^2/V(\phi_0)$. With these approximations, equation (35) can be used to derive $w(\Omega_\phi)$. The resulting expression for $w(\Omega_\phi)$ is identical to equation (23). Furthermore, numerical integration for the linear potential gives results in good agreement with our (w_0, w_a) fit discussed above (see Fig. 4 of Ref. [31]). This supports the conclusion that our results (equations 23 and 26) represent a generic asymptotic behavior. Of course, we cannot rule out the possibility of a more exact solution for the linear potential than the one we have outlined here.

IV. THE CASE AGAINST THAWING QUINTESSENCE

Now consider the arguments against the models considered here. To avoid unknown quantum gravity effects, it is desirable for the energy scale of the scalar field to be below the Planck mass (unity in our units). Requiring $\phi < 1$ does not constrain the models presented here, as the potential can be modified to shift the value of ϕ to any desired value. In Ref. [12], it was suggested that a possible constraint is $|V/V'| < 1$. Obviously, if this constraint is enforced, then all of the models considered here are ruled out, since we have only considered models with $V'/V < 1$ (equation 6). The implications of this proposed constraint are explored further in Ref. [13]. In Ref. [35], it was argued that the correct constraint is actually $\Delta\phi < 1$, where $\Delta\phi$ is the change in the value of ϕ from its initial value to the present. Our models do satisfy this second constraint.

Linder [13] has noted that thawing models with $\lambda_0 \ll 1$ occupy only a very small fraction of the phase plane defined by w and w'. This is certainly true; if one requires w to be very close to -1 at present, and assigns equal *a priori* weight to all phase trajectories in the $w-w'$ plane, then models with $\lambda_0 \ll 1$ are very unlikely.

Huterer and Peiris [36] performed Monte Carlo simulations of quintessence models, sampling low-order polynomial potentials. (See also the related work in Refs. [33, 34]). They found essentially no acceptable thawing models were generated using this procedure. This is not surprising, since their procedure samples a uniform distribution in the initial values of both $[(1/V)(dV/d\phi)]^2$ and $(1/V)(d^2V/d\phi^2)$, while our models require the ini-

tial values of both of these quantities to be much smaller than unity.

We do not dispute the conclusions in either Refs. [13] or [36]; they simply represent a different approach to determining the most plausible models. The models presented here require a very flat potential. One can argue that the special nature of the potential makes such models unlikely; however, we believe that an observed value of w near -1 argues in favor of choosing such special potentials, while the fact that all such models converge to a similar evolution makes these models more interesting. These thawing models do require a fine-tuning of ϕ_0; it must be chosen so that $V(\phi_0)$ is approximately equal to the dark energy density today.

The most serious problem with the models considered here is that there is currently no compelling observational evidence to favor them over a cosmological constant, as Fig. 5 shows. On the other hand, current observations do not rule out these thawing models.

V. CONCLUSIONS

Thawing models with potentials that satisfy the slow-roll conditions provide a natural way to produce w near -1, and they all converge to a single, universal behavior. Such models are, in some ways, the opposite of the tracker models proposed in Ref. [11]. The tracker models are insensitive to the initial conditions, but they depend sensitively on the shape of the potential over the entire range of evolution of ϕ. The models discussed here, in contrast, depend *only* on the initial conditions, i.e., the value of V and its derivatives at ϕ_0, but are insensitive to the shape of the rest of the potential. This situation arises because the field never rolls very far along the potential, and so never has a chance to "see" the rest of the potential.

These models provide a very well-defined form for $w(a)$ that depends only on the present-day values of w and Ω_ϕ. While we have provided a variety of arguments both for and against such models, it is obvious that these issues will ultimately be settled by observational data, rather than by the speculations of theorists like us.

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